Answers to the Exam
Random Geometry and Topology B

3 November 2025

Exercise 1 (20 pts)

The process $(\xi_t)_{t\geq 0}$ on the state space $S = \{0,1\}^{\mathbb{Z}}$ is defined by the following generator: for each $f : S \to \mathbb{R}$ cylinder function and for each $\eta \in S$

$$\Omega f(\eta) = \sum_{x\in\mathbb{Z}} \left(f(\eta^{0\to x}) - f(\eta)\right) + \sum_{\substack{x\in\mathbb{Z},\\ y\in\{x-1,x+1\}}} \lambda \left(f(\eta^{y\vee x\to x}) - f(\eta)\right) + \sum_{\substack{x\in\mathbb{Z},\\ y\in\{x-1,x+1\}}} \frac{1}{2}\left(f(\eta^{y\to x}) - f(\eta)\right)$$

with $\lambda > 0$ and local maps $\eta^{0\to x}, \eta^{y\vee x\to x}, \eta^{y\to x} : S \to S$ defined for each $x, y \in \mathbb{Z}$ as

$$\eta^{0\to x}(z) = \begin{cases} 0 & \text{if } z = x, \\ \eta(z) & \text{otherwise,} \end{cases} \qquad \eta^{y\vee x\to x}(z) = \begin{cases} \eta(y)\vee\eta(x) & \text{if } z = x, \\ \eta(z) & \text{otherwise,} \end{cases}$$

$$\eta^{y\to x}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(z) & \text{otherwise.} \end{cases}$$

We have seen in the lectures that if all the local maps defining the transition dynamics of an interacting particle system are monotone, then the interacting particle system is monotone as well. As $\eta^{0\to x}, \eta^{y\vee x\to x}, \eta^{y\to x}$ are monotone for all x and $y \in \mathbb{Z}$, the process is monotone.

Exercise 2 (20 pts)

For each $x \in \mathbb{Z}$ we define five Poisson processes that are independent from each other and from the Poisson processes of the other sites:

1. a rate 1 Poisson process, corresponding to the application of the map $\eta^{0\to x}$;
2. two rate λ Poisson processes, corresponding to the application of the maps $\eta^{x\pm1\vee x\to x}$;
3. two rate 1/2 Poisson processes, corresponding to the application of the maps $\eta^{x\pm1\to x}$.

For every x we place a *recovery mark* at (x,t) for each t that is an arrival time of the process of $\eta^{0\to x}$; we place a *type-1 arrow* from $(x\pm1,t)$ to (x,t) for each t that is an arrival time of the process of $\eta^{x\pm1\vee x\to x}$; and we place a *type-2 arrow* from $(x\pm1,t)$ to (x,t) for each t that is an arrival time of the process of $\eta^{x\pm1\to x}$.

A *path* from (x,s) to (y,t) in $\mathbb{Z}\times[0,\infty)$ with $s \leq t$ is a cadlag function $\gamma : [s,t] \to \mathbb{Z}$ satisfying

- **i)** $\gamma(s) = x$ and $\gamma(t) = y$;
- **ii)** $\forall u \in [s,t]$: $(\gamma(u), u)$ is not a recovery mark;
- **iii)** if $\gamma(u-) = \gamma(u)$ for some $u \in [s,t]$, then $(\gamma(u), u)$ is not the endpoint of a type-2 arrow.
- **iv)** if $\gamma(u-) \neq \gamma(u)$ for some $u \in [s,t]$, then there is an arrow from $(\gamma(u-), u)$ to $(\gamma(u), u)$.

That is, a path between (x,s) and (y,t) moves in the increasing time direction, does not go through recovery marks and the endpoints of type-2 arrows, and may jump along arrows (of any type).

We denote by

$$(x,s) \to (y,t) := \{\text{there is a path from } (x,s) \text{ to } (y,t)\}$$

and for $A, B \subseteq \mathbb{Z}$

$$(A,s) \to (B,t) := \{\exists x \in A, y \in B \text{ such that } (x,s) \to (y,t)\}.$$

Denoting by $(\xi_t^A)_{t\geq 0}$ the process in which initially every site in $A \subseteq \mathbb{Z}$ is infected and every other site is healthy we set

$$\xi_t^A := \{x \in \mathbb{Z} : (A,0) \to (x,t)\}.$$

Exercise 3 (20 pts)

We will couple the process $(\xi_t)_{t\geq 0}$ with a contact process $(\zeta_t)_{t\geq 0}$ with infection rate λ and recovery rate 2 using the graphical representation such that a.s.

$$\zeta_t \subseteq \xi_t \quad \forall t \geq 0.$$

We use the same graphical representation as above, but define infection paths for $(\zeta_t)_{t\geq 0}$ in the following way: an infection path between (x,s) and (y,t) moves in the increasing time direction, does not go through recovery marks and the endpoints of type-2 arrows, and may jump along type-1 arrows. (Informally, this can be interpreted as turning the endpoints of type-2 arrows into recovery marks for $(\zeta_t)_{t\geq 0}$.) As any infection path for $(\zeta_t)_{t\geq 0}$ is a valid path for $(\xi_t)_{t\geq 0}$, it is clear that the coupling has the desired properties. This implies

$$\mathbb{P}(\zeta_t^0 \neq \varnothing\ \forall t \geq 0) \leq \mathbb{P}(\xi_t^0 \neq \varnothing\ \forall t \geq 0).$$

Thus, if $\mathbb{P}(\zeta_t^0 \neq \varnothing\ \forall t \geq 0) > 0$, then $\mathbb{P}(\xi_t^0 \neq \varnothing\ \forall t \geq 0) > 0$ as well, yielding

$$\lambda_c^{2CP} \geq \lambda_c,$$

where λ_c^{2CP} is the critical parameter of $(\zeta_t)_{t\geq 0}$. It remains to show that $\lambda_c^{2CP} < \infty$. If we rescale the Poisson processes corresponding to $(\zeta_t)_{t\geq 0}$ by 2, then we get a contact process with infection rate $\lambda/2$ and recovery rate 1. The critical parameter of this process is then $\lambda_c^{CP} = \lambda_c^{2CP}/2$, and $\lambda_c^{CP} < \infty$ as seen in the lectures.

Exercise 4 (20 pts)

By definition, $(\xi_t)_{t\geq 0}$ and $(\tilde{\xi}_t)_{t\geq 0}$ are dual to each other with respect to the duality function $H : S \times \tilde{S} \to \mathbb{R}$, if for all $\xi_0 \in S$ and $\tilde{\xi}_0 \in \tilde{S}$

$$\mathbb{E}[H(\xi_0, \tilde{\xi}_t)] = \mathbb{E}[H(\xi_t, \tilde{\xi}_0)] \quad \forall t \geq 0.$$

As $H(\xi, \tilde{\xi}) = \mathbb{1}\{\xi \cap \tilde{\xi} = \varnothing\}$, this is equivalent to saying that for all $A, B \subset \mathbb{Z}$ with B finite the two processes satisfy

$$\mathbb{P}(A \cap \tilde{\xi}_t^B = \varnothing) = \mathbb{P}(\xi_t^A \cap B = \varnothing). \tag{1}$$

Using the graphical representation of $(\xi_t)_{t\geq 0}$ (from Ex. 2) we have $\xi_t^A = \{x \in \mathbb{Z} : (A,0) \to (x,t)\}$ for all $A \subseteq \mathbb{Z}, t \geq 0$. We define on the same graphical representation

$$\tilde{\xi}_t^B := \{x \in \mathbb{Z} : (x,0) \to (B,t)\} \quad \forall B \subset \mathbb{Z} \text{ finite}, t \geq 0.$$

This process indeed staisfies the definition of a dual process, as clearly

$$\{\xi_t^A \cap B \neq \varnothing\} = \{(A,0) \to (B,t)\} = \{\tilde{\xi}_t^B \cap A \neq \varnothing\}.$$

To determine the dynamics of $(\tilde{\xi}_t)_{t\geq 0}$ we turn the graphical representation upside-down: we reverse the direction of all arrrows and the direction of time and follow the paths backwards. The resulting picture is again the union of Poisson processes with the same rates as before. The backwards paths does not go through recovery marks, jump along type-2 arrows whenever they encounter (the starting point) of one and may jump along type-1 arrows. Therefore, $(\tilde{\xi}_t)_{t\geq 0}$ is a mixture of a system of coalescing random walks and a rate-λ contact process.

By definition, the upper invariant law $\bar{\nu}$ is the limiting distribution of the process started from the all-1 configuration (as $(\xi_t)_{t\geq 0}$ is monotone, this limit indeed exists). The density of $\bar{\nu}$ is thus

$$\overline{\nu}\{\eta \in S : 0 \in \eta\} = \lim_{t\to\infty} \mathbb{P}(0 \in \xi_t^{\mathbb{Z}}) = \lim_{t\to\infty} \mathbb{P}(\xi_t^{\mathbb{Z}} \cap \{0\} \neq \varnothing)$$

Letting $A = \mathbb{Z}, B = \{0\}$ in (1) we have

$$\mathbb{P}(\mathbb{Z} \cap \tilde{\xi}_t^0 \neq \varnothing) = \mathbb{P}(\xi_t^{\mathbb{Z}} \cap \{0\} \neq \varnothing).$$

Therefore,

$$\overline{\nu}\{\eta \in S : 0 \in \eta\} = \lim_{t\to\infty} \mathbb{P}(\mathbb{Z} \cap \tilde{\xi}_t^0 \neq \varnothing) = \lim_{t\to\infty} \mathbb{P}(\tilde{\xi}_t^0 \neq \varnothing) = \mathbb{P}(\tilde{\xi}_t^o \neq \varnothing\ \forall t \geq 0).$$

So indeed $\overline{\nu}\{\eta \in S : 0 \in \eta\} > 0$, if and only if $\mathbb{P}(\tilde{\xi}_t^o \neq \varnothing\ \forall t \geq 0) > 0$.

Exercise 5 (20 pts)

Let us denote by $(\bar{\xi}_t)_{t\geq 0}$ the modification of $(\xi_t)_{t\geq 0}$ in which infected sites cannot recover (that is, in the graphical representation it ignores recovery marks and type-2 arrows that would be used to adopt state 0). We can then couple $(\xi_t)_{t\geq 0}$ and $(\bar{\xi}_t)_{t\geq 0}$ in a way that a.s.

$$\xi_t \subseteq \bar{\xi}_t \quad \forall t \geq 0.$$

Denoting by $\bar{r}_t := \sup \bar{\xi}_t^0$ $(t \geq 0)$ this clearly implies that a.s.

$$r_t \leq \bar{r}_t \quad \forall t \geq 0.$$

Observe that $\bar{r}_0 = 0$ and $\bar{r}_t$ is non-decreasing in t. Furthermore, it increases by one whenever there is a type-1 or type-2 arrow from $(\bar{r}_t, t)$ to $(\bar{r}_t + 1, t)$. Therefore, the jump times of $\bar{r}_t$ correspond to the arrival times of a Poisson process with parameter $\lambda + 1/2$. The interarrival times $\tau_1, \tau_2, \ldots$ of this process are i.i.d. $\text{Exp}(\lambda + 1/2)$ random variabes. Using these observations we have

$$\begin{aligned}
\mathbb{P}_\lambda(r_t > Ct) &\leq \mathbb{P}_\lambda(\bar{r}_t > Ct) \\
&= \mathbb{P}(\tau_1 + \tau_2 + \cdots + \tau_{\lceil Ct\rceil} \leq t) && \text{pick } s > 0 \text{ small} \\
&= \mathbb{P}(e^{-s(\tau_1+\tau_2+\cdots+\tau_{\lceil Ct\rceil})} \geq e^{-st}) && \text{Markov inequality and } \tau_1, \tau_2, \ldots \text{ i.i.d.} \\
&\leq e^{st}\mathbb{E}(e^{-s\tau_1})^{\lceil Ct\rceil} && \mathbb{E}(e^{-s\tau_1}) < 1 \\
&\leq e^{st}\mathbb{E}(e^{-s\tau_1})^{Ct} && \text{M.G.F. of } \tau_1 \\
&= e^{st}\left(\frac{\lambda + \frac{1}{2}}{\lambda + \frac{1}{2} + s}\right)^{Ct}
\end{aligned}$$

We have $\frac{\lambda+1/2}{\lambda+1/2+s} = e^{-s'}$ for some small $s' > 0$. So if we pick $C > 0$ large enough such that $s - Cs' < 0$ and let $c = Cs' - s > 0$, then

$$\mathbb{P}_\lambda(r_t > Ct) \leq e^{(s-Cs')t} = e^{-ct}.$$